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THE SOLID ANGLE THEOREM IN RIGID BODY DYNAMICS†

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A supplement to MacCullagh's geometrical interpretation of the motion of a rigid body in the Euler case is given. It is shown that this supplement is exhausted by effects of a purely kinematic and geometrical nature.

According to Poinsot's geometrical interpretation, the inertial motion of a body around a fixed point can be represented as the rocking without sliding of its inertia ellipsoid over a fixed plane perpendicular to the kinetic-momentum vector. The centre of the inertia ellipsoid coincides with the centre of attachment of the body.

Poinsot's geometrical interpretation is not complete if we mean by a complete geometrical interpretation a clear representation of the sequence of positions of a rigid body in a space without indicating the instants of time at which these positions are reached. In other words, a complete geometrical interpretation is the interpretation of the trajectory in a configuration manifold (group $SO(3)$) without indicating the schedule of motion in it.

The incompleteness of Poinsot's geometrical interpretation is related to the condition for there to be no slipping and can be illustrated by the example of the plane motion of an oval disc. Suppose that at a certain instant of time a point A of the disc is in contact with the horizontal straight line along which rocking occurs without slipping. The problem of at what point on the straight line a certain other point of the disc B will come into contact with it requires an additional operation of integration without which the sequence of positions of the disc in the plane cannot be stated. The clarity of Poinsot's geometrical interpretation should not introduce an error: it is related to the illusion of complete determinacy of rocking without slipping in everyday life.

There is one other geometrical interpretation of the motion of a body in the Euler case, proposed by MacCullagh [1].

MacCullagh starts from the first two integrals written in projections of the kinetic momentum onto an axis connected with the body. The integral of the kinetic energy has the form

$$x^2/A + y^2/B + z^2/C = 2T$$

which is the equation of an ellipsoid in x, y, z axes

$$(x = Ap, \quad y = Bq, \quad z = Cr).$$

In these axes the integral of the angular momentum is the equation of a sphere: $x^2 + y^2 + z^2 = K^2$.

The kinetic-energy ellipsoid is called a gyrational ellipsoid. Its intersection with the kinetic-momentum sphere forms trajectories (kinetic polhodes), along which the end of the kinetic-momentum vector in the body moves. Since the kinetic-momentum vector is fixed in space, the motion of the body can be represented as the rolling of a connected gyrational ellipsoid of this fixed vector along the kinetic polhode. The kinetic momentum then describes a cone in the body, which is its generatrix . The directrix of this cone is also a kinetic polhode (Fig. 1).

MacCullagh's geometrical interpretation is also incomplete. The position of the rigid body around the axis of kinetic momentum remains undetermined.

Recently, Montgomery [2] attempted to supplement MacCullagh's geometrical interpretation and, starting from the equations of the dynamics of a rigid body, calculated the angle which the body swings around the axis of kinetic momentum after this axis describes a closed conical surface in the body. This angle was equal to the projection of the angular velocity of the body onto the kinetic-momentum vector (this projection is independent of time), multiplied by the time for a complete rotation, plus the solid angle of the cone.

Montgomery's supplement of the geometrical interpretation is partial, since it only completely defines the position of the body at an instant of time that is a multiple of the period.

Nevertheless, Montgomery's results can be generalized considerably and MacCullagh's geometrical interpretation can be made exhaustive.

We have the following theorem.

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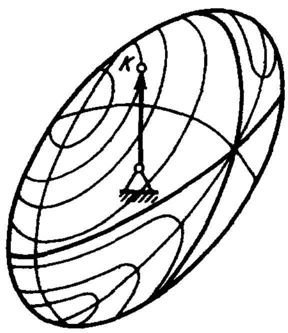

Fig. 1.

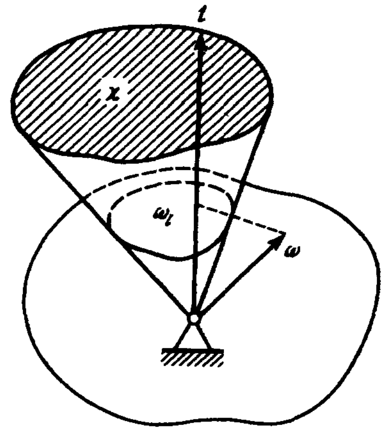

Fig. 2.

Theorem 1. If, when a rigid body with one fixed point is rotating, a certain straight line **l**, fixed in space, describes a closed conical surface in the body, the angle of rotation of the body around this axis is equal to the integral of the projection of the angular velocity onto this axis minus the solid angle of the cone described (Fig. 2)

$$\gamma = \int_0^T \omega_l(t)dt - \chi$$

This is a generalization of Montgomery's result, which consists of the fact that it is not limited to the Euler case and holds for any motions of a rigid body, whatever the momenta applied to the body.

Moreover, it enables one, as will be shown below, to give a complete description of the position of a rigid body at any instant of time (at any point of the polhode), and not only after a period.

Before we prove this theorem we note that a theorem related to Theorem 1 was formulated and proved in 1952 by A. Yu. Ishlinskii.

Theorem 2 (Ishlinskii's theorem). If, when a body with a single fixed point is rotating, a certain straight line **l**, fixed in the body, describes a closed conical surface in space, the angle of rotation of the body around this axis is equal to the integral of the projection of the angular velocity of the body onto this axis plus the solid angle of the cone described

$$\gamma = \int_0^T \omega_l(t)dt + \chi$$

These theorems are related since the first deals with the angular velocity of the body in a projection onto an axis fixed in space, while the second deals with the projection onto an axis fixed in the body.

Theorem 2 bears no direct relation to the problem of a geometrical interpretation of the Euler case being discussed, but Theorem 1 follows from it. The proof of Theorem 2, unlike the extremely complicated result proved by Montgomery, is completely elementary.

Proof of Theorem 2. We rigidly connect with the body a triangle such that the third axis is directed along the vector **l**. In Krylov–Bulgakov angles we have $\omega_l = \gamma + \alpha \sin\beta$, whence

$$\gamma = -\oint \sin\beta d\alpha + \int_0^T \omega_l(t)dt$$

Using Green's theorem for a curvilinear integral, we obtain a surface integral of an element of area on the unit sphere, which also proves the assertion.

Proof of Theorem 1. We change the point of view: the body rotating in space will be assumed to be fixed; and the fixed space will be assumed to rotate around the body. Then, we can apply Theorem 2 to the space. Assuming that the angle of rotation of the body relative to space differs in sign from the angle of rotation of space about the body, we obtain the required result.

Both theorems are a special case of the following Theorem 3 [4].

Theorem 3. If, when a rigid body with one fixed point is rotating, a certain straight line describes a closed conical surface both in the body and in the space, the angle of rotation of the body around its axis is equal to the integral of the projection of the angular velocity onto this axis plus the difference in the solid angles described by this axis in the body and in space.

We will now consider the problem of supplementing MacCullagh's geometrical interpretation, using Theorem 1.

Suppose the vertical axis **l**, fixed in space, describes an open surface in the body, in which motion along the kinetic polhode begins from the point O and ends at the point A (Fig. 3). We complete the cone to a closed surface by plane rotation by an angle θ. When the position of the trihedron, rigidly connected to the body, at the instant when the axis **l** passes through the point A (the current position of the rigid body), relative to the initial position of this trihedron, connected to the point O, can be defined by two angles: the angle of rotation around the axis perpendicular to the plane containing the initial and current position of the axis **l** in the body, and the angle γ of rotation of the body around the axis **l**. The latter equals the angle defined by Theorem 1 for the closed cone completed by the method described.

Note that this method is impossible when constructing Montgomery's proof, since plane rotation around an arbitrary axis in the body is not in general, inertial motion.

It is easy to see, that in all three theorems discussed, the angle of rotation of the body of interest to us consists of two terms of quite different meaning. The first term is the integral of the projection of the angular velocity on the axis **l**, i.e. the purely kinematic component of the rotation. The second term, equal to the solid angle, has a purely geometrical origin. It is the effect of parallel transfer of the vector along a closed contour on a unit Riemann sphere. Suppose an axis, rigidly connected to the body, traces out a closed curve on the unit sphere (Fig. 4). Then, the condition for the projection of the angular velocity of the trihedron **l**, **a**, **l** × **a** onto the axis I to be zero is also the condition for parallel transfer of the vector **a**. We know from Riemannian geometry that the angle of rotation of the vector **a** is determined by the Gaussian curvature of the manifold, and, in the case of the unit sphere, is equal to the area of the spherical segment enclosed inside the curve.

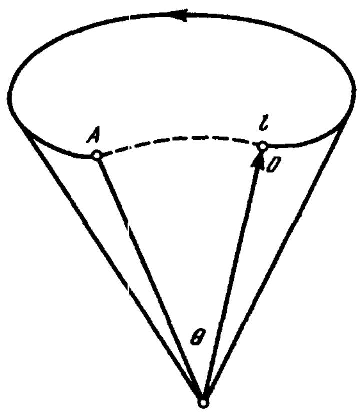

Fig. 3.

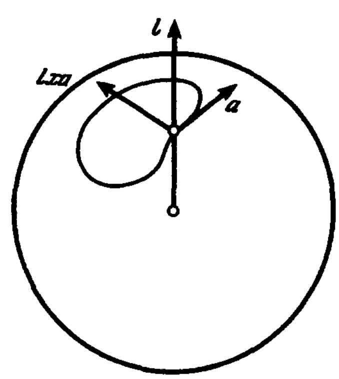

Fig. 4.

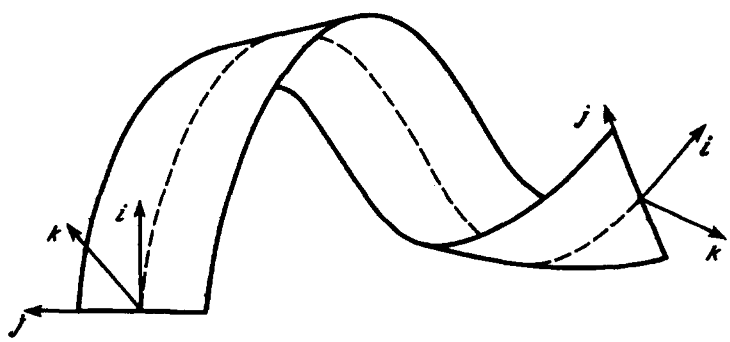

Fig. 5.

In this connection it would be of interest to investigate the mechanical meaning of the effect of a parallel transfer of the vector along a closed contour on the manifold of positions of a rigid body with one fixed point (group $SO(3)$).

In conclusion we note that the application of the solid-angle theorems formulated above is outside the framework not only of Euler's case but also of rigid-body dynamics in general. These theorems can be used in all cases when the problem reduces to analysing the mutual angular position of two trihedra.

For example, suppose that the mean length of a thin inextensible tape, which is absolutely flexible in one direction and absolutely rigid in the other, is bent into a certain spatial curve (Fig. 5). The orientation of the initial section of the tape with respect to the end section can be determined as the orientation of trihedra connected with these sections. If the initial trihedron is shifted along the mean line so that the vector **i** touches the median line, and the vector **k** is situated along the normal to the tape, the projection of the angular velocity of the trihedron onto the **k** axis is zero (the condition for absolute rigidity of the tape in the **i**, **j** plane) and the answer to the problem in question is given by Theorem 2, in which $\omega_r(t) \equiv 0$.

One other example is the problem of the transmission of a polarized light beam through a lightguide bent in space. The problem of the rotation of the polarization plane also reduces to a solid-angle theorem.

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